FREE CURVES AND PERIODIC POINTS FOR TORUS HOMEOMORPHISMS

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Abstract. We study the relationship between free curves and periodic points for torus homeomorphisms in the homotopy class of the identity. By free curve we mean a homotopically nontrivial simple closed curve that is disjoint from its image. We prove that every rational point in the rotation set is realized by a periodic point provided that there is no free curve and the rotation set has empty interior. This gives a topological version of a theorem of Franks. Using this result, and inspired by a theorem of Guillou, we prove a version of the Poincaré-Birkhoff Theorem for torus homeomorphisms: in the absence of free curves, either there is a fixed point or the rotation set has nonempty interior.

1. Introduction

Given a torus homeomorphism $F\colon \mathbb{T}^d \to \mathbb{T}^d$ in the identity homotopy class, the *rotation set* of a lift $f\colon \mathbb{R}^d \to \mathbb{R}^d$ of F was introduced by Misiurewicz and Ziemian in [MZ89], and it is defined as the set of all accumulation points of sequences of the form

$$\left\{\frac{f^{n_i}(x_i) - x_i}{n_i}\right\}_{i\in\mathbb{N}}$$

where $n_i \to \infty$ and $x_i \in \mathbb{R}^2$.

This set carries dynamical information about F, but understanding this information is not as easy as in the one-dimensional case. Moreover, when $d = 2$ the rotation set has nice geometric properties (for example, convexity) which are no longer valid in higher dimension. For this reason, we restrict our attention to the two-dimensional setting.

A central problem, inspired by the Poincaré theory for circle homeomorphisms, is to determine when a point $(p_1/q, p_2/q) \in \rho(f) \cap \mathbb{Q}^2$ (with $\gcd(p_1, p_2, q) = 1$) is realized by a periodic orbit of F, i.e. when can we find a point $x \in \mathbb{R}^2$ such that

$$f^q(x) = x + (p_1, p_2).$$

This problem has been thoroughly studied, especially by Franks, who proved in [Fra88] that extremal points of the rotation set are always realized by periodic orbits. However, it is generally not true that *every* rational point in the rotation set is realized. In this aspect, the case that is best understood is when the rotation set has non-empty interior. In fact, Franks proved in [Fra89] that every rational point in the interior of the rotation set is realized by a periodic orbit, and this is optimal in the sense that there are examples (even area-preserving ones) where the rotation set has non-empty interior and many rational points on the boundary, but

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the only ones that are realized by periodic orbits are extremal or interior ones (see [MZ91, §3]).

On the other hand, when the rotation set has empty interior the situation is more delicate. It is easy to construct a periodic-point-free homeomorphism $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ such that its rotation set is a segment containing many rational points. However, this cannot happen under some additional hypothesis:

Theorem 1.1 (Franks [Fra95])**.** *If an area-preserving homeomorphism of $\mathbb{T}^2$ in the homotopy class of the identity has a rotation set with empty interior, then every rational point in its rotation set is realized by a periodic orbit.*

In the same article, Franks asks whether the area-preserving hypothesis is really necessary for the conclusion of the theorem. It is natural to expect that a weaker, more topological hypothesis should suffice to obtain the same result. This topological substitute for the area-preserving hypothesis turns out to be, to some extent, the *curve intersection property.* An essential simple closed curve is *free* for F if $F(\gamma) \cap \gamma = \emptyset$. We say that F has the curve intersection property if F has no free curves. Our first result can be stated as follows:

Theorem A. *If a homeomorphism of $\mathbb{T}^2$ in the homotopy class of the identity satisfies the curve intersection property and its rotation set has empty interior, then every rational point in its rotation set is realized by a periodic orbit.*

Our proof is essentially different of that of Franks' theorem, since the latter relies strongly on chain-recurrence properties that are guaranteed by the area preserving hypothesis but not by the curve intersection property.

Variations of the curve intersection property are already present in some fixed point theorems. An interesting case is a generalization of the classic theorem of Poincaré-Birkhoff [Bir25], which states that for an area-preserving homeomorphism of the closed annulus, isotopic to the identity and verifying the *boundary twist condition*, there exist at least two fixed points. Birkhoff and Kerékjártó already noted that, for getting at least one fixed point, the area preserving hypothesis was not completely necessary, and they replaced it by the weaker condition that any essential simple closed curve intersects its image by the homeomorphism [Ker29].

An even more topological version of this theorem was proved by Guillou, who substituted the twist condition by the property that every simple arc joining the boundary components intersects its image by F:

Theorem (Guillou, [Gui94])**.** *If $F\colon [0,1]\times S^1 \to [0,1]\times S^1$ is an orientation-preserving homeomorphism isotopic to the identity and such that every essential simple closed curve or simple arc joining boundary components intersects its image by F, then F has a fixed point.*

The hypotheses of the above theorem can be regarded as the curve intersection property in the setting of the closed annulus. This led Guillou to ask if a similar result holds for the torus. The answer is no, as an example by Bestvina and Handel shows [BH92]. Their example relies in the fact that the existence of a free curve imposes a restriction on the "size" of the rotation set (see Lemma 3.4). For this reason, their example has a rotation set with nonempty interior, which implies that it has infinitely many periodic points of arbitrarily high periods [Fra89], and positive topological entropy [LM91]. The question that naturally arises is whether

the presence of this kind of "rich" dynamics is the only new obstruction to the existence of free curves in $\mathbb{T}^2$. In other words, *is the answer to Guillou's question affirmative if the rotation set has empty interior?* This leads to our next result.

Theorem B. *Let $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism homotopic to the identity and satisfying the curve intersection property. Then either F has a fixed point, or its rotation set has nonempty interior.*

Thus, if F has the curve intersection property, then either F has periodic orbits of arbitrarily high periods or it has a fixed point. In the latter case, one might expect the existence of a second fixed point (as in the case of the annulus) but no more than that. Figure 1 shows that one cannot expect more than two fixed points in the annulus; the time-one map of the flow sketched there is a homeomorphism with the curve intersection property, which has two fixed points and no other periodic points. Gluing a symmetric copy of this homeomorphism through the boundaries of the annulus, one obtains an example with the same properties in $\mathbb{T}^2$.

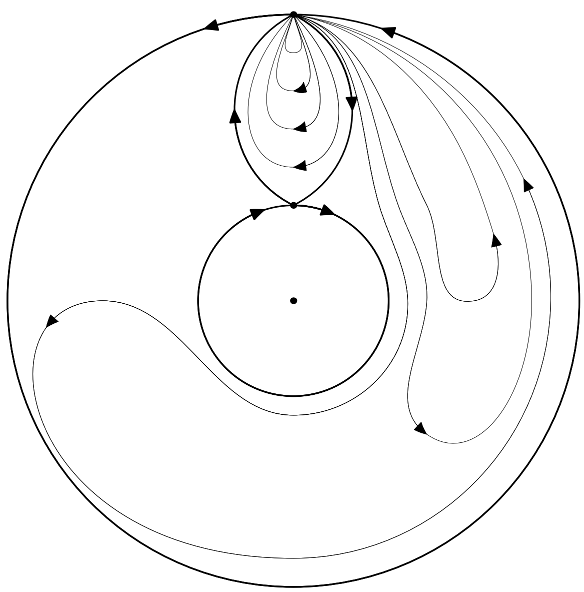


FIGURE 1. A curve-intersecting flow with no periodic points of period > 1

Conjecture 1.2. *Under the hypotheses of Theorem B, either F has two fixed points or F has periodic points of arbitrarily high periods.*

2. Notation and preliminaries

As usual, we denote the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ by $\mathbb{T}^2$, being $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$ the canonical quotient projection. We denote the integer translations by

$$T_1\colon (x, y) \mapsto (x + 1, y) \text{ and } T_2\colon (x, y) \mapsto (x, y + 1),$$

and $\mathrm{pr}_i\colon \mathbb{R}^2 \to \mathbb{R}$, for $i = 1, 2$ are the projections onto the first and second coordinate, respectively.

By $\mathrm{Homeo}(X)$ we mean the set of homeomorphisms of X onto itself, and by $\mathrm{Homeo}_*(X)$ the set of elements of $\mathrm{Homeo}(X)$ which are homotopic to the identity.

Given $F \in \mathrm{Homeo}(\mathbb{T}^2)$, a lift of F to $\mathbb{R}^2$ is a map $f \in \mathrm{Homeo}(\mathbb{R}^2)$ such that $\pi f = F\pi$. A homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of an element of $\mathrm{Homeo}_*(\mathbb{T}^2)$ if and only if f commutes with T_1 and T_2. Any two lifts of a given homeomorphism of $\mathbb{T}^2$ always differ by an integer translation. We will usually denote maps of $\mathbb{T}^2$ to itself by uppercase letters, and their lifts to $\mathbb{R}^2$ by their corresponding lowercase letters.

By $\mathbb{Z}^2_{\mathrm{cp}}$ we denote the set of pairs of integers (m, n) such that m and n are coprime. We will say that $(x, y) \in \mathbb{R}^2$ is an integer point if both x and y are integer, and a rational point if both x and y are rational. When we write a rational number as p/q, we assume that it is in reduced form, i.e. that p and q are coprime, except when we are talking about a rational point $(p_1/q, p_2/q)$, in which case we assume that p_1, p_2 and q are mutually coprime (i.e. $\gcd\{p_1, p_2, q\} = 1$).

2.1. **The rotation set.** From now on, we will assume that $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$, and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of F.

Definition 2.1. (Misiurewicz & Ziemian [MZ89]) The *rotation set of f* is defined as

$$\rho(f) = \bigcap_{m=1}^{\infty} \mathrm{cl}\left(\bigcup_{n=m}^{\infty}\left\{\frac{f^n(x) - x}{n} : x \in \mathbb{R}^2\right\}\right) \subset \mathbb{R}^2$$

The *rotation set of a point* $x \in \mathbb{R}^2$ is defined by

$$\rho(f, x) = \bigcap_{m=1}^{\infty} \mathrm{cl}\left\{\frac{f^n(x) - x}{n} : n > m\right\}.$$

If the above set consists of a single point v, we call v the rotation vector of x.

Remark 2.2. It is easy to see that for integers n, m_1, m_2,

$$\rho(T_1^{m_1} T_2^{m_2} f^n) = n\,\rho(f) + (m_1, m_2).$$

In particular, the rotation set of any other lift of F is an integer translate of $\rho(f)$, and we can talk about the "rotation set of F" if we keep in mind that it is defined modulo $\mathbb{Z}^2$.

Theorem 2.3 ([MZ89])**.** *The rotation set is compact and convex, and every extremal point of $\rho(f)$ is the rotation vector of some point.*

Given $A \in \mathrm{GL}(2, \mathbb{Z})$, we denote by $\tilde{A}$ the homeomorphism of $\mathbb{T}^2$ lifted by it. If $H \in \mathrm{Homeo}(\mathbb{T}^2)$, there is a unique $A \in \mathrm{GL}(2, \mathbb{Z})$ such that for every lift h of H, the map $h - A$ is bounded (in fact, $\mathbb{Z}^2$-periodic). From this it follows that H is isotopic to $\tilde{A}$, and $h^{-1} - A^{-1}$ is bounded. In fact, $H \mapsto A$ induces an isomorphism between the isotopy group of $\mathbb{T}^2$ and $\mathrm{GL}(2, \mathbb{Z})$.

Lemma 2.4. *Let $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$, $A \in \mathrm{GL}(2, \mathbb{Z})$, and $H \in \mathrm{Homeo}(\mathbb{T}^2)$ isotopic to $\tilde{A}$. Let f and h be the respective lifts of F and H to $\mathbb{R}^2$. Then*

$$\rho(hfh^{-1}) = A\,\rho(f).$$

In particular, $\rho(AfA^{-1}) = A\,\rho(f)$.

Proof. We can write $((hfh^{-1})^n(x) - x)/n$ as

$$\frac{(h - A)(f^n h^{-1}(x))}{n} + A\left(\frac{f^n(h^{-1}(x)) - h^{-1}(x)}{n}\right) + A\left(\frac{(h^{-1} - A^{-1})(x)}{n}\right)$$

and using the fact that $h - A$ and $h^{-1} - A^{-1}$ are bounded, we see that the leftmost and rightmost terms of the above expression vanish when $n \to \infty$. Thus if $n_k \to \infty$ and $x_k \in \mathbb{R}^2$, we have

$$\lim_{k\to\infty} \frac{(hfh^{-1})^{n_k}(x_k) - x_k}{n_k} = A\left(\lim_{k\to\infty} \frac{f^{n_k}(h^{-1}(x_k)) - h^{-1}(x_k)}{n_k}\right)$$

whenever the limits exist. Since h is a homeomorphism, it follows from the definition that $\rho(hfh^{-1}) = A\,\rho(f)$. □

We will use the above lemma extensively: when trying to prove some property that is invariant by topological conjugacy (like the existence of a free curve or a periodic point for F), it allows us to consider just the case where the rotation set is the image of $\rho(f)$ by some convenient element of $\mathrm{GL}(2, \mathbb{Z})$.

Remark 2.5. A particular case that will often appear is when $\rho(f)$ is a segment of rational slope. In that case, there exists a map $A \in \mathrm{GL}(2, \mathbb{Z})$ such that $A\,\rho(f)$ is a vertical segment. Indeed, if $\rho(f)$ is a segment of slope p/q, then we can find $x, y \in \mathbb{Z}$ such that $px + qy = 1$, and letting

$$A = \begin{pmatrix} p & -q \\ y & x \end{pmatrix}$$

it follows that $\det(A) = 1$. Since $A(q, p) = (0, 1)$, $A\,\rho(f)$ is vertical.

2.2. **The rotation set and periodic orbits.** Recall that we say that a rational point $(p_1/q, p_2/q) \in \rho(f)$ is realized by a periodic orbit if there is $x \in \mathbb{R}^2$ such that

$$f^q(x) = x + (p_1, p_2).$$

As we already mentioned in the introduction, a rational point in the rotation set is not necessarily realized by a periodic orbit. However, we have the following "realization" results (including the already mentioned Theorem 1.1).

Theorem 2.6 (Franks [Fra88])**.** *If a rational point of $\rho(f)$ is extremal, then it is realized by a periodic orbit.*

Theorem 2.7 (Franks [Fra89])**.** *If a rational point is in the interior of $\rho(f)$, then it is realized by a periodic orbit.*

Theorem 2.8 (Jonker & Zhang [JZ98])**.** *If $\rho(f)$ is a segment with irrational slope and it contains a point of rational coordinates, then this point is realized by a periodic orbit.*

We thank the referee for bringing the following generalization of the above theorem to our attention. It is stated for diffeomorphisms in [LC91, p. 106], but its proof remains valid for homeomorphisms using the results of [LC05] (see p. 9 of that article).

Theorem 2.9 (Le Calvez)**.** *If a rational point of $\rho(f)$ belongs to a line of irrational slope which bounds a closed half-plane that contains $\rho(f)$, then this point is realized by a periodic orbit.*

2.3. **Curves and lines.** We denote by I the interval $[0,1]$. A *curve* on a manifold M is a continuous map $\gamma\colon I \to M$. As usual, we represent by γ both the map and its image, as it should be clear from the context which is the case.

We say that the curve γ is *closed* if $\gamma(0) = \gamma(1)$, and *simple* if the restriction of the map γ to the interior of I is injective. If γ is a closed curve, we say it is *essential* if it is homotopically non-trivial.

Definition 2.10. A curve $\gamma \subset M$ is *free* for F if $F(\gamma) \cap \gamma = \emptyset$. We say that F has the *curve intersection property* if there are no free essential simple closed curves.

Remark 2.11. For convenience, from now on by a *free curve* for F we will usually mean an essential simple closed curve that is free for F, unless otherwise stated.

By a *line* we mean a proper topological embedding $\ell\colon \mathbb{R} \to \mathbb{R}^2$. Again, we use ℓ to represent both the function and its image. Note that lines are oriented.

Definition 2.12. Given $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$, a (p,q)-line in $\mathbb{R}^2$ is a line ℓ such that there exists $\tau > 0$ such that $\ell(t+\tau) = T_1^p T_2^q \ell(t)$ for all $t \in \mathbb{R}$, and such that its projection to $\mathbb{T}^2$ by π is a simple closed curve. A (p,q)-curve in $\mathbb{T}^2$ is the projection by π of a (p,q)-line. We will say that a simple closed curve is *vertical* if it is either a $(0,1)$-curve or a $(0,-1)$-curve. Similarly, a line will be called vertical if it is a $(0,1)$-line or a $(0,-1)$-line.

Remark 2.13. If ℓ is a line in $\mathbb{R}^2$ that is invariant by $T_1^p T_2^q$, then $\pi(\ell)$ is always a closed curve in $\mathbb{T}^2$. We are requiring that this curve be simple to call ℓ a (p,q)-line. Conversely, if γ is an essential simple closed curve in $\mathbb{T}^2$, taking a lift $\tilde{\gamma}\colon I \to \mathbb{R}^2$, we have that $(p,q) = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ is an integer point independent of the choice of the lift. The curve $\tilde{\gamma}$ can be extended naturally to $\mathbb{R}$ by $\tilde{\gamma}(t+n) = \tilde{\gamma}(t) + n(p,q)$, if $n \in \mathbb{Z}$ and $t \in [0,1]$; in this way we obtain a (p,q)-line that projects to γ. Since γ is simple and essential, it is not hard to see that p and q must be coprime. With some abuse of notation, we will say that the (p,q)-line ℓ is a lift of $\pi(\ell)$.

Remark 2.14. Note that any (p,q)-line is contained in a strip bounded by two straight lines of slope q/p. In particular, if ℓ is a vertical line, there is $M > 0$ such that $\mathrm{pr}_1(\ell) \subset [-M, M]$.

Given a line ℓ in $\mathbb{R}^2$, there are exactly two connected components of $\mathbb{R}^2 \setminus \ell$. Using the orientation of ℓ, we may define the *left* and the *right* components, which we denote by $L\ell$ and $R\ell$. We also denote by $\overline{L}\ell$ and $\overline{R}\ell$ their respective closures, which correspond to $L\ell \cup \ell$ and $R\ell \cup \ell$.

Given two lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$, we write $\ell_1 < \ell_2$ if $\ell_1 \subset L\ell_2$ and $\ell_2 \subset R\ell_1$. With an abuse of notation we will write $\ell_1 \leq \ell_2$ when $\ell_1 \subset \overline{L}\ell_2$, which means that the lines may intersect but only "from one side". We will mostly use this relation to compare (p,q)-lines of the same type. Note that if ℓ_1 and ℓ_2 are disjoint (p,q)-lines, then either $\ell_1 < \ell_2$ or $\ell_2 < \ell_1$ (but not both). In this context, we denote by $S(\ell_1, \ell_2)$ the strip $L\ell_2 \cap R\ell_1$, and by $\overline{S}(\ell_1, \ell_2)$ its closure.

Remark 2.15. If $f \in \mathrm{Homeo}(\mathbb{R}^2)$ preserves orientation, then f preserves order: if $\ell_1 < \ell_2$, then $f(\ell_1) < f(\ell_2)$.

2.4. **Brouwer lines.** A Brouwer line for $h \in \mathrm{Homeo}_*(\mathbb{R}^2)$ is a line ℓ in $\mathbb{R}^2$ such that $\ell < h(\ell)$ (sometimes Brouwer lines are not assumed to be oriented, but our lines are oriented). The classic Brouwer Translation Theorem guarantees the existence of a Brouwer line through any point of $\mathbb{R}^2$ for any fixed-point free, orientation-preserving homeomorphism (see [Bro12, Ker29, Fat87, Fra92, Gui94]; also see [LC05] for a powerful equivariant version). The following will be much more useful for our purposes:

Theorem 2.16 (Guillou [Gui06])**.** *Let f be a lift of an orientation-preserving $F \in \mathrm{Homeo}(\mathbb{T}^2)$, and suppose f has no fixed points. Then f has a Brouwer (p,q)-line, for some $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$.*

The following lemma is particularly useful when there is a Brouwer (p,q)-line:

Lemma 2.17. *Let S be a closed semiplane determined by a straight line containing the origin, and for $y \in \mathbb{R}^2$ denote by $S_y = \{w+y : w \in S\}$ its translate by y. Suppose that $x \in \mathbb{R}^2$ is such that for some y,*

$$f^n(x) \in S_y \text{ for all } n > 0.$$

Then $\rho(f,x) \subset S$. Moreover, if for all $x \in \mathbb{R}^2$ there is y such that the above holds, then $\rho(f) \subset S$.

Proof. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear functional such that

$$S_y = \{w \in \mathbb{R}^2 : \phi(w) \geq \phi(y)\}.$$

Given x such that $f^n(x) \in S_y$ for all $n > 0$, we have then

$$\phi\left(\frac{f^n(x) - x}{n}\right) = \frac{\phi(f^n(x)) - \phi(x)}{n} \geq \frac{\phi(y) - \phi(x)}{n} \to 0.$$

Thus, if z is the limit of a sequence of the form $(f^{n_i}(x) - x)/n_i$, then $\phi(z) \geq 0$. This implies that $\rho(f,x) \subset S$. The other claim follows from Theorem 2.3. ☐

Remark 2.18. If f has a Brouwer (p,q)-line ℓ, the above lemma and Remark 2.14 imply that $\rho(f)$ is contained in one of the closed semiplanes determined by the straight line L of slope q/p through the origin. Indeed, by Remark 2.14, we can choose one of those semiplanes S such that $S_y \supset R\ell \supset S_{y'}$ for some $y, y' \in \mathbb{R}^2$. Given $k \in \mathbb{N}$, we know that $\ell < f^k(\ell)$, so that $S_y \supset f^k(R\ell) \supset f^k(S_{y'})$. Given $x \in \mathbb{R}^2$, there is $(m,n) \in \mathbb{Z}^2$ such that $x + (m,n) \in S_{y'}$, and it follows $f^k(x) \in S_{y-(m,n)}$ for all $k \in \mathbb{N}$. Thus by the above lemma, $\rho(f) \subset S$.

2.5. **The rotation set and free curves.** Besides the existence of periodic orbits, other practical dynamical information that can be obtained from the rotation set is the existence of free curves. Recall that an interval with rational endpoints $[p/q, p'/q']$ is a Farey interval if $qp' - pq' = 1$. The following result was proved by Kwapisz for diffeomorphisms, and by Beguin, Crovisier, LeRoux and Patou for homeomorphisms.

Theorem 2.19 ([Kwa02], [BCLP04])**.** *Suppose there exists a Farey interval $[p/q, p'/q']$ such that*

$$\mathrm{pr}_1\, \rho(f) \subset \left(\frac{p}{q}, \frac{p'}{q'}\right).$$

Then there exists a simple closed $(0,1)$*-curve* γ *in* $\mathbb{T}^2$ *such that all the curves* γ, $F(\gamma)$, $F^2(\gamma)$, $\ldots$, $F^{q+q'-1}(\gamma)$ *are mutually disjoint. In particular, if* $\operatorname{pr}_1 \rho(f) \cap \mathbb{Z} = \emptyset$*, then* F *has a free* $(0,1)$*-curve.*

2.6. **The wedge.** We now define an operation between (p,q)-lines, which is a fundamental tool in the proof of Theorem A.

Recall that a Jordan domain is an open topological disk bounded by a simple closed curve, and recall the following

Theorem 2.20 (Kerékjártó, [Ker23])**.** *If* U_1 *and* U_2 *are two Jordan domains in the two-sphere, then each connected component of* $U_1 \cap U_2$ *is a Jordan domain.*

Identifying $\mathbb{R}^2 \cup \{\infty\}$ with the two-sphere $\mathbb{S}^2$, we may regard lines in $\mathbb{R}^2$ as simple closed curves in $\mathbb{S}^2$ containing ∞. Recall that any two (p,q)-lines are oriented in the same way; that is, the intersection of their left sides is not contained in a strip. Thus if ℓ_1 and ℓ_2 are (p,q)-lines, it is easy to see that $L\ell_1 \cap L\ell_2$ has a unique connected component containing ∞ in its boundary. By the above theorem, this boundary is a simple closed curve, so that it corresponds to a line in $\mathbb{R}^2$. One can easily see that this new line is also a (p,q)-line, if oriented properly. This motivates the following

Definition 2.21. Given two (p,q)-lines ℓ_1 and ℓ_2 in $\mathbb{R}^2$, their *wedge* $\ell_1 \wedge \ell_2$ is the line defined as the boundary of the unique unbounded connected component of $L\ell_1 \cap L\ell_2$, oriented so that this component corresponds to $L(\ell_1 \wedge \ell_2)$.

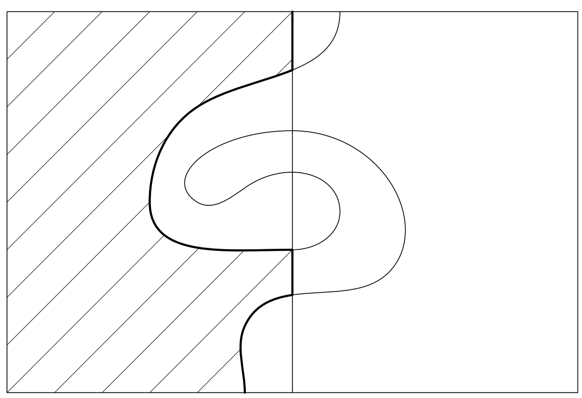


Figure 2. The wedge of two lines

Remark 2.22. This operation is called 'join' in [BCLP04] and denoted by $\vee$.

We denote the wedge of multiple lines $\ell_1, \ldots, \ell_n$ by

$$\ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_n = \bigwedge_{i=1}^{n} \ell_i.$$

This is well defined because the wedge is commutative and associative. The following proposition resumes the interesting properties of the wedge.

Proposition 2.23. *The wedge is commutative, associative and idempotent. Furthermore,*

(1) *The wedge of (p,q)-lines is a (p,q)-line;*
(2) *If $h \in \mathrm{Homeo}(\mathbb{R}^2)$ is a lift of a torus homeomorphism, then $h(\ell_1 \wedge \ell_2) = h(\ell_1) \wedge h(\ell_2)$;*
(3) *$\ell_1 \wedge \ell_2 \leq \ell_1$ and $\ell_1 \wedge \ell_2 \leq \ell_2$;*
(4) *If $\ell_1 < \ell_2$ and $\xi_1 < \xi_2$, then $\ell_1 \wedge \xi_1 < \ell_2 \wedge \xi_2$;*
(5) *The wedge of Brouwer lines is a Brouwer line.*

3. Realizing periodic orbits

In this section we prove Theorem A. From now on we assume that $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$ and f is a lift of F. The following result will be essential in the proof.

Proposition 3.1. *Suppose F^n has a free (p,q)-curve for some $n \geq 1$. Then F has a free (p,q)-curve.*

We will also use the following lemmas, the proofs of which are postponed to the end of this section.

Lemma 3.2. *Suppose f^n has a Brouwer $(0,1)$-line, for some $n \in \mathbb{N}$. Then f has a Brouwer $(0,1)$-line.*

Lemma 3.3. *Suppose that some lift f of F has a Brouwer $(0,1)$-line. Then, either F has a free $(0,1)$-curve, or $\max(\mathrm{pr}_1 \rho(f)) \geq 1$.*

The next lemma is essentially Lemma 3 of [BH92]; we include it here for the sake of completeness.

Lemma 3.4. *Suppose F has a free $(0,1)$-curve. Then for any lift f of F there is $k \in \mathbb{Z}$ such that $\mathrm{pr}_1 \rho(f) \subset [k, k+1]$.*

3.1. **Proof of Proposition 3.1.** Conjugating the involved maps by an element of $\mathrm{GL}(2,\mathbb{Z})$, we may assume that $(p,q) = (0,1)$ (see Lemma 2.4 and the remark below it).

Suppose F^n has a free $(0,1)$-curve for some $n \geq 2$. Then, any lift to $\mathbb{R}^2$ of this curve, with a possible reversion of orientation, is a vertical Brouwer line for f^n. Thus, Lemma 3.2 implies that there is a vertical Brouwer line for f. This holds for any lift f of F.

If $\mathrm{pr}_1 \rho(f) \cap \mathbb{Z} = \emptyset$, then by Theorem 2.19 there is a free $(0,1)$-curve for F, and we are done.

Otherwise, let $k \in \mathrm{pr}_1 \rho(f) \cap \mathbb{Z}$, and consider the lift $f_0 = T_1^{-k} f$ of F. By Remark 2.2, it is clear that $0 \in \mathrm{pr}_1 \rho(f_0)$. On the other hand, as we already saw, f_0 has a vertical Brouwer line ℓ.

Assume ℓ is a Brouwer $(0,1)$-line. Then by Lemma 3.3, either F has a free $(0,1)$-curve, or $\max(\mathrm{pr}_1 \rho(f_0)) \geq 1$. In the latter case, since we know that $0 \in \rho(f_0)$, it follows from connectedness that $\mathrm{pr}_1 \rho(f_0) \supset [0,1]$. But this implies that

$$\mathrm{pr}_1 \rho(f_0^n) \supset [0,n] \supset [0,2],$$

which contradicts Lemma 3.4 (since F^n has a free $(0,1)$-curve). Thus the only possibility is that F has a free $(0,1)$-curve.

If ℓ is a Brouwer $(0,-1)$-line, then using the previous argument with f_0^{-1} instead of f_0, we see that F^{-1} (and thus F) has a free $(0,-1)$-curve γ; and inverting the orientation of γ we get a free $(0,1)$-curve for F. This completes the proof. $\square$

3.2. **Proof of Theorem A..** Suppose $F \in \mathrm{Homeo}_*(\mathbb{T}^2)$ has the curve intersection property and $\rho(f)$ has empty interior, where f is a lift of F. We have three cases.

3.2.1. $\rho(f)$ *is a single point.* In this case, the unique point of $\rho(f)$ is extremal; and if it is rational, Theorem 2.6 implies that it is realized by a periodic orbit of F.

3.2.2. $\rho(f)$ *is a segment of irrational slope.* In this case $\rho(f)$ contains at most one rational point and, by Theorem 2.8, this point is realized by a periodic orbit.

Remark 3.5. Theorem 2.16 provides a simple way of proving this as well. In fact, it suffices to consider the case where the unique rational point in $\rho(f)$ is the origin, and to show that in this case f has a fixed point. If the origin is an extremal point, this follows from Theorem 2.6. If the origin is strictly inside the rotation set, then there is only one straight line through the origin such that $\rho(f)$ is contained in one of the closed semiplanes determined by the line. This unique line is the one with the same slope as $\rho(f)$, which is irrational. If f has no fixed points, then Theorem 2.16 implies that f has a Brouwer (p,q)-line for some $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$; but then our previous claim contradicts Remark 2.18.

3.2.3. $\rho(f)$ *is a segment of rational slope.* Fix a rational point $(p_1/q, p_2/q) \in \rho(f)$. Recall that this point is realized as the rotation vector of a periodic orbit of F if and only if $g = T_1^{-p_1}T_2^{-p_2}f^q$ has a fixed point. Note that $(0,0) \in \rho(g)$, and g is a lift of F^q. Moreover,

$$\rho(g) = T_1^{-p_1}T_2^{-p_2}(q \cdot \rho(f)),$$

which is a segment of rational slope containing the origin. Conjugating g by an element of $\mathrm{GL}(2,\mathbb{Z})$, we may assume that $\rho(g)$ is a vertical segment containing the origin (c.f. Remark 2.5).

We will show by contradiction that g has a fixed point. Suppose this is not the case. Then, by Theorem 2.16, g has a Brouwer (p,q)-line ℓ, for some $(p,q) \in \mathbb{Z}^2_{\mathrm{cp}}$. Moreover, $(0,0)$ must be strictly inside $\rho(g)$ (i.e. it cannot be extremal, since otherwise g would have a fixed point by Theorem 2.6), and by Remark 2.18 this implies that ℓ is a vertical Brouwer line.

Assume ℓ is a $(0,1)$-line (if it is a $(0,-1)$-line, we may consider g^{-1} instead of g and use a similar argument). Since $\mathrm{diam}(\mathrm{pr}_1\,\rho(g)) = 0$, Lemma 3.3 implies that F^q, the map lifted by g, has a free $(0,1)$-curve; but then by Proposition 3.1, F has a free curve, contradicting the curve intersection property. This concludes the proof. $\square$

3.3. **Proof of Lemma 3.2.** Let ℓ be a Brouwer $(0,1)$-line for f^n, for some $n > 1$. We will show that there is a Brouwer $(0,1)$-line for f^{n-1}; by induction, it follows that there is a Brouwer $(0,1)$-line for f.

We know that $\ell < f^n(\ell)$. Let ξ be a $(0,1)$-line such that $\ell < \xi < f^n(\ell)$, and define

$$\ell' = \xi \wedge \bigwedge_{i=1}^{n-1} f^i(\ell).$$

By Proposition 2.23, ℓ' is still a $(0,1)$-line. We claim that it is a Brouwer line for f^{n-1}. In fact,

$$\begin{aligned} f^{n-1}(\ell') &= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{n-1}(f^i(\ell)) \\ &= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{i-1}(f^n(\ell)) \\ &= f^{n-1}(\xi) \wedge f^n(\ell) \wedge \bigwedge_{i=2}^{n-1} f^{i-1}(f^n(\ell)) \\ &= f^{n-1}(\xi) \wedge f^n(\ell) \wedge \bigwedge_{i=1}^{n-2} f^{i}(f^n(\ell)). \end{aligned}$$

Using the facts that

$$f^{n-1}(\ell) < f^{n-1}(\xi), \quad \xi < f^n(\ell), \text{ and } f^i(\ell) < f^i(f^n(\ell)),$$

and Proposition 2.23, we see that

$$\ell' = \xi \wedge \bigwedge_{i=1}^{n-1} f^i(\ell) = f^{n-1}(\ell) \wedge \xi \wedge \bigwedge_{i=1}^{n-2} f^i(\ell) < f^{n-1}(\ell'),$$

so that ℓ' is a Brouwer $(0,1)$-line for f^{n-1}. This concludes the proof.

3.4. **Proof of Lemma 3.3.** Let ℓ be a Brouwer $(0,1)$-line for f. We consider two cases.

3.4.1. *Case 1. For all $n > 0$, $f^n(\ell) \not< T_1^n(\ell)$.* In this case, for each $n > 0$ we can choose $x_n \in \ell$ such that $f^n(x_n) \in \overline{R}(T_1^n\ell)$. From the fact that ℓ is a $(0,1)$-line we also know that $\mathrm{pr}_1(\ell) \subset [-M, M]$ for some $M > 0$, and therefore

$$\mathrm{pr}_1(T_1^n(\ell)) \subset [-M+n, M+n].$$

Hence,

$$\mathrm{pr}_1(f^n(x_n)) \in \mathrm{pr}_1(R(T_1^n(\ell))) \subset [-M+n, \infty).$$

It then follows that

$$\mathrm{pr}_1\left(\frac{f^n(x_n) - x_n}{n}\right) \geq \frac{(-M+n) - M}{n} = -2\frac{M}{n} + 1 \xrightarrow{n\to\infty} 1,$$

and by the definition of rotation set this implies that some point $(x,y) \in \rho(f)$ satisfies $x \geq 1$; i.e. $\max \mathrm{pr}_1(\rho(f)) \geq 1$.

3.4.2. *Case 2. $f^n(\ell) < T_1^n\ell$ for some $n > 0$.* We will show that in this case F has a free $(0,1)$-curve. The idea is similar to the proof of Lemma 3.2. Let n be the smallest positive integer such that $f^n(\ell) < T_1^n\ell$. If $n = 1$, we are done, since $\ell < f(\ell) < T_1\ell$, so that ℓ projects to a free $(0,1)$-curve for F.

Now assume $n > 1$. We will show how to construct a new Brouwer $(0,1)$-line β for f such that $f^{n-1}(\beta) < T_1^{n-1}\beta$. Repeating this argument $n-1$ times, we end up with a $(0,1)$-line ℓ' such that $\ell' < f(\ell') < T_1(\ell')$, so that ℓ' projects to a free curve, completing the proof.

Let ξ be a $(0,1)$-curve such that

$$f^n(\ell) < \xi < T_1^n\ell \tag{1}$$

We may choose ξ such that $\xi < f(\xi)$, by taking it close enough to $f^n(\ell)$. This is possible because $f^n(\ell) < f(f^n(\ell))$, and these two curves are separated by a positive distance, since they both project to $(0,1)$-curves in $\mathbb{T}^2$. Thus ξ is also a Brouwer $(0,1)$-line for f.

Define

$$\beta = \xi \wedge \bigwedge_{i=1}^{n-1} T_1^{n-i} f^i(\ell).$$

Let us see that $f^{n-1}(\beta) < T_1^{n-1}(\beta)$. Since T_1 commutes with f, we have

$$\begin{aligned}
f^{n-1}(\beta) &= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{n-1} T_1^{n-i} f^i(\ell) \\
&= f^{n-1}(\xi) \wedge \bigwedge_{i=1}^{n-1} f^{i-1} T_1^{n-i} f^n(\ell) \\
&= f^{n-1}(\xi) \wedge T_1^{n-1} f^n(\ell) \wedge \bigwedge_{i=2}^{n-1} f^{i-1} T_1^{n-i} f^n(\ell)
\end{aligned}$$

By (1), we also have

- $f^{n-1}(\xi) < f^{n-1}(T_1^n \ell)$,
- $T_1^{n-1} f^n(\ell) < T_1^{n-1}\xi$, and
- $f^{i-1} T_1^{n-i} f^n(\ell) < f^{i-1} T_1^{n-i} T_1^n \ell$.

Using these facts and Proposition 2.23 we see that

$$\begin{aligned}
f^{n-1}(\beta) &< f^{n-1}(T_1^n \ell) \wedge T_1^{n-1}\xi \wedge \bigwedge_{i=2}^{n-1} f^{i-1} T_1^{n-i} T_1^n(\ell) \\
&= T_1^{n-1}(T_1 f^{n-1}(\ell)) \wedge T_1^{n-1}(\xi) \bigwedge_{i=2}^{n-1} T_1^{n-1}(T_1^{n-(i-1)} f^{i-1}(\ell)) \\
&= T_1^{n-1}\left(\xi \wedge T_1 f^{n-1}(\ell) \wedge \bigwedge_{i=1}^{n-2} T_1^{n-i} f^i(\ell)\right) \\
&= T_1^{n-1}\left(\xi \wedge \bigwedge_{i=1}^{n-1} T_1^{n-i} f^i(\ell)\right) \\
&= T_1^{n-1}(\beta).
\end{aligned}$$

Since β is a Brouwer $(0,1)$-line, this completes the proof. □

3.5. **Proof of Lemma 3.4.** Let f be a lift of F, and suppose that F has a free $(0,1)$-curve. A lift of this curve is a $(0,1)$-line ℓ such that $T_1^n f(\ell) \cap \ell = \emptyset$ for any $n \in \mathbb{N}$. In particular, given $n \in \mathbb{N}$, either $\ell < T_1^n f(\ell)$ or $T_1^n f(\ell) < \ell$ (and in the latter case reversing the orientation of ℓ we get a Brouwer $(0,-1)$-line for $T_1^n f$). In either case, Remark 2.18 implies that $\rho(T_1^n f)$ is contained in one of the semiplanes $\{(x,y) : x \geq 0\}$ or $\{(x,y) : x \leq 0\}$, and by Remark 2.2 this means that $\rho(f)$ is contained in one of the semiplanes $\{(x,y) : x \geq -n\}$ or $\{(x,y) : x \leq -n\}$. Thus n cannot be an interior point of $\mathrm{pr}_1\, \rho(f)$. Since this holds for all n, and $\mathrm{pr}_1\, \rho(f)$ is an interval, there is k such that $\mathrm{pr}_1\, \rho(f) \subset [k, k+1]$. □

4. Free curves and fixed points

In this section we prove Theorem B. For the proof, we have two main cases. The first one is when the rotation set is either a segment of rational slope or a single point; this is dealt with Theorem A and the results stated in §2. The second case is when the rotation set is a segment of irrational slope. In that case, the main idea is to find $A \in \mathrm{GL}(2,\mathbb{R})$ such that $A\,\rho(f)$ has no integers in the first or second coordinate, so that we may apply directly Theorem 2.19 to AfA^{-1} (c.f. Lemma 2.4). In fact, using this argument we obtain the following more general result:

Theorem 4.1. *Suppose $\rho(f)$ is a segment of irrational slope with no rational points. Then for each $n > 0$ there is an essential simple closed curve γ such that γ, $F(\gamma), \dots, F^n(\gamma)$ are pairwise disjoint.*

Remark 4.2. No example is known of a homeomorphism that meets the hypothesis of the above theorem. In fact, if a conjecture of Franks and Misiurewicz turns out to be true (see [FM90]), then no such example exists. This theorem could be a (very small) step towards this conjecture.

The problem of finding the map A previously mentioned is mainly arithmetic, and we consider it first. In the next section, we briefly recall a few facts about continued fractions that will be needed in the proof; in §4.2 we prove the two arithmetic lemmas that allow us to find the map A; in §4.3 we prove Theorem 4.1; finally, in §4.4 we complete the proof of Theorem B.

4.1. **Continued fractions.** Given an integer a_0 and positive integers $a_1, \dots, a_n$, we define

$$[a_0; a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}.$$

Given $\alpha \in \mathbb{R}$, define $\{\alpha_n\}$ and $\{a_n\}$ recursively by $a_0 = \lfloor \alpha \rfloor$, $\alpha_0 = \alpha - a_0$, and

$$a_{n+1} = \left\lfloor \alpha_n^{-1} \right\rfloor,\ \alpha_{n+1} = \alpha_n^{-1} - a_{n+1},$$

whenever $\alpha_n \neq 0$. This gives the continued fractions representation of α: If α is rational, we get a finite sequence $a_0, \dots, a_n$, and

$$\alpha = [a_0; a_1, \dots, a_n].$$

If α is irrational, then the sequence is infinite and

$$\alpha = [a_0; a_1, a_2, \dots] \doteq \lim_{n\to\infty} [a_0; a_1, \dots, a_n].$$

The rational number $p_n/q_n = [a_0; a_1 \dots, a_n]$ is called the n-th *convergent* to α. Convergents may be regarded as the "best rational approximations" to α, in view of the following properties (see, for instance, [HW90])

Proposition 4.3. *If p_n/q_n are the convergents to α, then*

(1) *$\{q_n\}$ is an increasing sequence of positive integers, and*

$$\frac{1}{q_n + q_{n+1}} < (-1)^n(\alpha q_n - p_n) < \frac{1}{q_{n+1}}.$$

(2) $\frac{p_{2n}}{q_{2n}} < \frac{p_{2n+2}}{q_{2n+2}} < \alpha < \frac{p_{2n+3}}{q_{2n+3}} < \frac{p_{2n+1}}{q_{2n+1}}$.

(3) $p_{n+1}q_n - p_nq_{n+1} = (-1)^n$

A Farey interval is a closed interval with rational endpoints $[p/q, p'/q']$ such that $p'q - q'p = 1$. Note that two consecutive convergents give Farey intervals.

Proposition 4.4. *Let $[p/q, p'/q']$ be a Farey interval. Then*

$$\left[\frac{p+p'}{q+q'}, \frac{p'}{q'}\right] \quad \text{and} \quad \left[\frac{p}{q}, \frac{p+p'}{q+q'}\right]$$

are Farey intervals, and if $p''/q'' \in (p/q, p'/q')$, then $q'' \geq q + q'$.

4.2. **Arithmetic lemmas.** Define the vertical and horizontal inverse *Dehn twists* by

$$D_1\colon (x,y) \mapsto (x-y, y), \;\; D_2\colon (x,y) \mapsto (x, y-x).$$

Let Q be the set of vectors of $\mathbb{R}^2$ with positive coordinates; for $u = (x,y) \in Q$, we denote by $\mathrm{slo}(u) = y/x$ the *slope* of u. Let Q_1 and Q_2 be the sets of elements of Q having slope smaller than one and greater than one, respectively.

Remark 4.5. Note the following simple properties

(1) for $i = 1, 2$, $D_iQ_i = Q$; and if $u \in Q$, then $D_i^{-k}u \in Q_i$ for all $k > 0$;
(2) for $i = 1, 2$, $\|D_iu\| < \|u\|$ if $u \in Q_i$;
(3) $\mathrm{slo}(D_2^k u) = \mathrm{slo}(u) - k$ and $\mathrm{slo}(D_1^k u)^{-1} = \mathrm{slo}(u)^{-1} - k$.

Lemma 4.6. *Let u and v be elements of Q with different slopes. Then there is $A \in \mathrm{GL}(2,\mathbb{Z})$ such that*

(1) *$\|Au\| \leq \|u\|$;*
(2) *$\|Av\| \leq \|v\|$;*
(3) *Both Au and Av are in Q, and either one of these points is on the diagonal, or one is in Q_1 and the other in Q_2.*

Proof. We first note that it suffices to consider the case where both u and v are in Q_1. Indeed, if one of the vectors is in Q_i and the other is not (for $i = 1$ or 2), we can set $A = Id$; and if both u and v are in Q_2 then we may use Su and Sv instead, where S is the isometry $(x,y) \mapsto (y,x)$.

Given $u \in Q_1$, we define a sequence of matrices $A_n \in \mathrm{SL}(2,\mathbb{Z})$ and integers a_n by $A_0 = I$, $a_0 = 0$, and recursively (see Figure 3)

- If $\mathrm{slo}(A_nu) = 1$ stop the construction.
- a_{n+1} is the smallest integer such that $D_i^{a_{n+1}}A_nu \notin Q_i$, where $i = 2$ if n is odd, 1 if n is even;
- $A_{n+1} = D_i^{a_{n+1}}A_n$.

In this way we get either an infinite sequence, or a finite sequence $A_1, \dots A_N$ such that A_Nu lies on the diagonal and has positive coordinates. Furthermore, given $0 \leq n < N$ if the sequence is finite, or $n \geq 0$ if it is infinite, we have

(1) $A_nu \in Q_i$ where $i = 2$ if n is odd, 1 if n is even;
(2) If $\alpha_n = \mathrm{slo}(A_nu)^{(-1)^n}$, then $\alpha_n = \alpha_{n-1}^{-1} - a_n$
(3) $\|A_0u\|, \|A_1u\|, \dots$ is a decreasing sequence;

The first property is a consequence of the definition. The second follows from Remark 4.5, since for odd n, we have

$$\alpha_n = \mathrm{slo}(D_1^{a_n}A_{n-1}u)^{-1} = \mathrm{slo}(A_{n-1}u)^{-1} - a_n = \alpha_{n-1}^{-1} - a_n$$

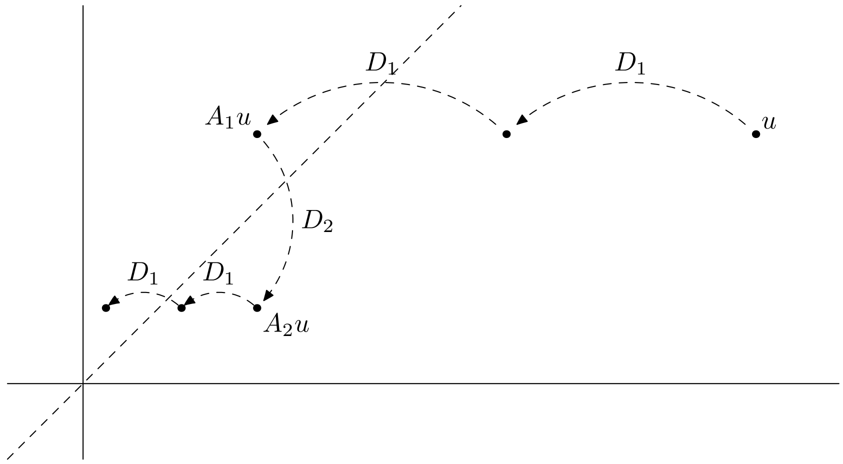


FIGURE 3. The sequence A_iu

while for even n,

$$\alpha_n = \mathrm{slo}(D_2^{a_n}A_{n-1}u) = \mathrm{slo}(A_{n-1}u) - a_n = \alpha_{n-1}^{-1} - a_n.$$

The last property also follows from the construction: if $A_nu = D_i^{a_n}A_{n-1}u$, then $D_i^kA_{n-1}u \in Q_i$ for all $0 \leq k < a_n$, so that Remark 4.5 implies that $\|A_nu\| < \|A_{n-1}u\|$.

If n is odd, a_n is the smallest integer such that $D_1^{a_n}A_{n-1}u \notin Q_1$, or equivalently (assuming that $n < N$ if the sequence a_n is finite), the smallest integer such that

$$\mathrm{slo}(D_1^{a_n}A_{n-1}u) = (\mathrm{slo}(A_{n-1}u)^{-1} - a_n)^{-1} = (\alpha_{n-1}^{-1} - a_n)^{-1} \geq 1.$$

Since $A_{n-1}u \in Q_1$, $\mathrm{slo}(A_{n-1}u) < 1$ so that $\alpha_{n-1}^{-1} > 1$, and $a_n > 0$. Note that α_n^{-1} cannot be an integer, since otherwise $\mathrm{slo}(A_nu) = 1$, which contradicts the fact that $A_nu \in Q_2$; thus

$$a_n = \left\lfloor \alpha_{n-1}^{-1} \right\rfloor.$$

If n is even, the above equation holds by a similar argument. One easily sees from these facts that $\{\alpha_n\}$ coincides with the sequence obtained in the definition of the continued fractions expression of $\alpha_0 = \mathrm{slo}(u)$, and thus $\{a_n\}$ coincides with the continued fractions coefficients of $\mathrm{slo}(u)$.

Now given $v \in Q_1$ with $\mathrm{slo}(v) \neq \mathrm{slo}(u)$, define in the same way as above sequences of positive integers b_n and matrices $B_n \in \mathrm{SL}(2, \mathbb{Z})$ such that $B_0 = I$, b_n is given by the continued fractions expression of $\mathrm{slo}(v)$, and $B_{n+1} = D_i^{b_{n+1}}B_n$ where $i = 2$ if n is odd, 1 if n is even. As before, we have that $\|B_nv\|$ is a (finite or infinite) decreasing sequence, and if it is finite of length N then $\mathrm{slo}(B_Nv) = 1$. Also $B_nv \in Q_i$ where $i = 2$ if n is odd, 1 if n is even (given that $n < N$ if the sequence is finite).

Since u and v have different slopes, the continued fractions expressions of these slopes cannot coincide. Thus there exists $m \geq 0$ such that $a_0 = b_0, \ldots, a_m = b_m$, and either $b_{m+1} \neq a_{m+1}$, or exactly one of a_{m+1} or b_{m+1} is not defined. In the latter case, we may assume that a_{m+1} is undefined (by swapping u and v if necessary) and this means that $\mathrm{slo}(A_mu) = 1$ (from the previous construction), so that $A = A_m = B_m$ meets the required conditions. In the former case, we may further assume that $a_{m+1} < b_{m+1}$ (again, by swapping u and v if necessary). This

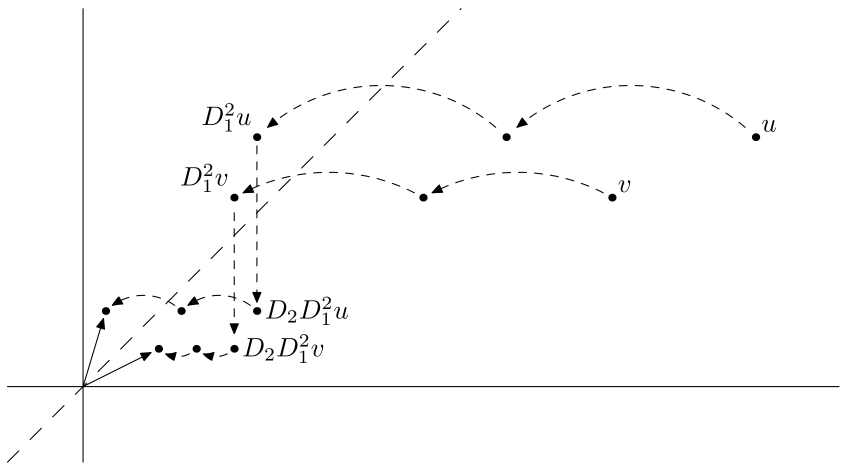


FIGURE 4. Example

means that $A_k = B_k$ for $0 \leq k \leq m$; so that if m is even, $A_m u \in Q_1$ and $A_m v \in Q_1$, but since $a_{m+1} < b_{m+1}$, it holds that

$$A_{m+1}u = D_1^{a_{m+1}} A_m u \notin Q_1 \text{ but } A_{m+1}v = D_1^{a_{m+1}} B_m v \in Q_1;$$

that is, $D_1^{a_{m+1}}$ "pushes" $A_m u$ out of Q_1, while leaving $A_m v$ in Q_1 (see Figure 4).

Moreover, since a_{m+1} is minimal with that property, either $\operatorname{slo}(A_{m+1}u) = 1$ or $A_{m+1}u \in Q_2$; and in either case $A_{m+1}u$ has positive coordinates. By construction, it also holds that $\|A_{m+1}v\| \leq \|v\|$ and $\|A_{m+1}u\| \leq \|u\|$.

If m is odd, a similar argument holds, and we see that $A_{m+1}u \in Q_2$ while $A_{m+1}v$ is either on the diagonal or in Q_1.

Starting from u and v in Q_1 we obtained $A = A_{m+1} \in \mathrm{GL}(2, \mathbb{Z})$ such that Au and Av have positive coordinates and either one of them is on the diagonal or one is in Q_1 and the other in Q_2; furthermore, $\|Au\| \leq \|u\|$ and $\|Av\| \leq \|v\|$. This completes the proof. □

Lemma 4.7. *Let $w = (x, y)$ be a vector with irrational slope. Then, for any $\epsilon > 0$ there exists $A \in \mathrm{SL}(2, \mathbb{Z})$ such that $\|Aw\| < \epsilon$.*

Proof. Let p_i/q_i be the convergents to y/x, and define

$$A_k = \begin{pmatrix} (-1)^k p_k & (-1)^{k+1} q_k \\ p_{k+1} & -q_{k+1} \end{pmatrix}.$$

By Proposition 4.3 we have that $\det A = 1$, the sequence $q_1, q_2, \ldots$ is increasing, and

$$\left| p_i - \frac{y}{x} q_i \right| < \frac{1}{q_{i+1}}$$

for all $i \geq 1$. Hence,

$$|\mathrm{pr}_1 A_k w| = \left| x \left(p_k - \frac{y}{x} q_k \right) \right| < \frac{|x|}{q_{k+1}},$$

and similarly

$$|\mathrm{pr}_2 A_k w| = \left| x \left(p_{k+1} - \frac{y}{x} q_{k+1} \right) \right| < \frac{|x|}{q_{k+2}};$$

Choosing k large enough so that $q_{k+1} > \sqrt{2}\,|x|\,\epsilon^{-1}$, we have $\|A_k w\| < \epsilon$. □

4.3. **Proof of Theorem 4.1.** We divide the proof into two cases:

4.3.1. *The case* $n = 1$. . We first assume $n = 1$, i.e. we prove that F has a free curve, assuming that $\rho(f)$ is a segment of irrational slope containing no rational points. The problem is reduced, by Lemma 2.4 and Theorem 2.19, to finding $A \in \mathrm{GL}(2,\mathbb{Z})$ such that the projection of $A\,\rho(f)$ to the first or the second coordinate contains no integers. Note that Lemma 4.7 allows us to assume that

$$\mathrm{diam}(\rho(f)) < \epsilon < \frac{1}{2\sqrt{5}}.$$

We may also assume that there are $m_1 \in \mathrm{pr}_1(\rho(f)) \cap \mathbb{Z}$ and $m_2 \in \mathrm{pr}_2(\rho(f)) \cap \mathbb{Z}$, for otherwise there is nothing to do.

Then using $T_1^{-m_1} T_2^{-m_2} f$ (which also lifts F) instead of f, we have that the extremal points of $\rho(f)$ are in opposite quadrants. By conjugating f with a rotation by $\pi/2$, we may assume that $\rho(f)$ is the segment joining $u = (-u_1, -u_2)$ and $v = (v_1, v_2)$ where $v_i \geq 0$ and $u_i \geq 0$, $i = 1, 2$. From this, and the fact that $\mathrm{diam}(\rho(f)) < \epsilon$, it follows that

$$\|u\| < \epsilon, \ \text{ and } \ \|v\| < \epsilon.$$

Case 1. One of the points has a zero coordinate. It is clear that neither u nor v can have both coordinates equal to 0. Conjugating by an appropriate isometry in $\mathrm{GL}(2,\mathbb{Z})$, we may assume the generic case that $u = (-u_1, 0)$, with $u_1 > 0$. Then $v_2 > 0$: in fact if $v_2 = 0$ then $\rho(f)$ contains the origin, which is not possible. Let $k > 0$ be the greatest integer such that $\mathrm{pr}_1 D_1^k v > -u_1$, i.e.

$$k = \left\lfloor \frac{u_1 + v_1}{v_2} \right\rfloor.$$

Note that $D_1^k u = u$, so that $D_1^k\,\rho(f)$ is the segment joining u to $D_1^k v$ (see Figure 5a). Moreover, $D_1^{k+1} v = (v_1', v_2)$ where $v_1' = v_1 - (k+1)v_2 < -u_1$. Thus

$$\max \mathrm{pr}_1(D_1^{k+1}\,\rho(f)) = -u_1 < 0.$$

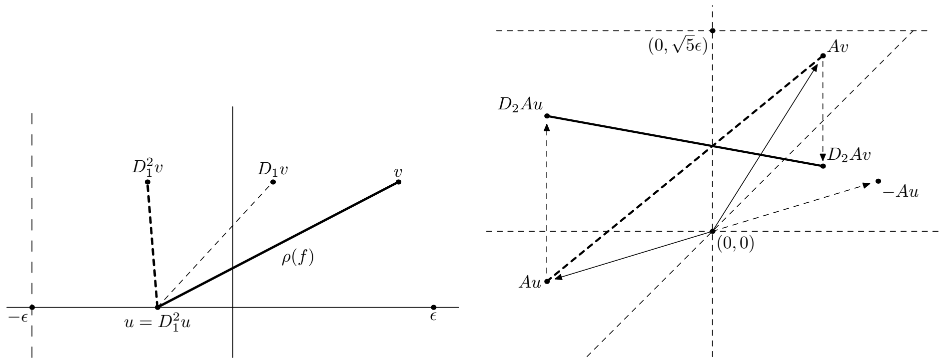
On the other hand,

$$\min \mathrm{pr}_1(D_1^{k+1}\,\rho(f)) \geq -u_1 - v_1 > -2\epsilon > -1,$$

so that taking $A = D_1^{k+1}$ we have $\mathrm{pr}_1(\rho(AfA^{-1})) \subset (-1, 0)$. By Theorem 2.19, it follows that $\tilde{A} F \tilde{A}^{-1}$ has a free $(0,1)$ curve. Thus F has a free curve.

Case 2. None of the points has a zero coordinate. In this case, $u \in -Q$ and $v \in Q$. Since the segment joining u to v cannot contain the origin, $-u$ and v are elements of Q with different slopes; thus Lemma 4.6 implies that there is $A \in \mathrm{GL}(2,\mathbb{Z})$ such that $\|Au\| \leq \|u\|$, $\|Av\| \leq \|v\|$, both $-Au$ and Av are in Q, and either one of them lies on the diagonal, or they are in opposite sides of the diagonal. This means that Au and Av are both contained in one of the closed semiplanes determined by the diagonal. By using $-A$ instead of A if necessary, we may assume that both are in the closed semiplane above the diagonal, which is mapped by D_2 to the upper semiplane $H = \{(x, y) : y \geq 0\}$. Note that

$$\|D_2 A u\| \leq \|D_2\|\,\|Au\| \leq \sqrt{5}\,\|u\|,$$



a) in the first coordinate b) in the second coordinate

FIGURE 5. Avoiding integers

and similarly $\|D_2 Av\| \leq \sqrt{5}\,\|v\|$. If $\mathrm{pr}_2(D_2Av) > 0$ and $\mathrm{pr}_2(D_2Au) > 0$, then (see Figure 5b) we have that

$$\mathrm{pr}_2\,\rho(f) \subset (0, \sqrt{5}\epsilon) \subset (0,1),$$

and by Theorem 2.19 (as in Case 1) it follows that F has a free curve. On the other hand, if either of D_2Av or D_2Au has zero second coordinate, the argument in Case 1 implies that F has a free curve.

This completes the proof when $\rho(f)$ has irrational slope and $n = 1$.

4.3.2. *The case $n > 1$.* Note that when $\rho(f)$ has irrational slope, $\rho(f^n) = n\,\rho(f)$ has irrational slope for all n.

Let $N = n!$. As we saw in the previous case, conjugating our maps by some $A \in \mathrm{GL}(2,\mathbb{Z})$, we may assume $\mathrm{pr}_1\,\rho(f^N) \cap \mathbb{Z} = \emptyset$; thus

$$\mathrm{pr}_1\,\rho(f^N) \subset (K, K+1) \text{ for some } K \in \mathbb{Z},$$

and by Remark 2.2,

$$\mathrm{pr}_1\,\rho(f) \subset \left(\frac{K}{N}, \frac{K+1}{N}\right).$$

Let $[p/q, p'/q']$ be the smallest Farey interval containing $\mathrm{pr}_1(\rho(f))$. We claim that $q + q' > n$. In fact, if $q + q' \leq n$, then $[K/N, (K+1)/N]$ must be contained in one of the smaller Farey intervals (see Proposition 4.4)

$$\left[\frac{p}{q}, \frac{p+p'}{q+q'}\right] \quad \text{or} \quad \left[\frac{p+p'}{q+q'}, \frac{p'}{q'}\right].$$

This is because $N(p+p')/(q+q')$ is an integer, so that it cannot be in the interior of $[K, K+1]$. But we chose our Farey interval to be the smallest, so $[p/q, p'/q']$ must as well be contained in one of these two intervals, which is a contradiction. Thus $q + q' > n$, and Theorem 2.19 guarantees that there is an essential simple closed curve γ such that its first n iterates by F are pairwise disjoint. This completes the proof.

4.4. **Proof of Theorem B..** Assume that $\rho(f)$ has empty interior. We will show that either F has a fixed point, or it has a free curve. There are several cases:

- $\rho(f)$ is a segment of irrational slope which contains no rational points. Then there is a free curve, by Theorem 4.1.
- $\rho(f)$ is a segment of irrational slope containing a rational non-integer point $(p_1/q, p_2/q)$. By Lemma 4.7 there exists $A \in \mathrm{GL}(2,\mathbb{Z})$ such that $\mathrm{diam}(A\,\rho(f)) < 1/q$. One of the two coordinates of $A(p_1/q, p_2/q)$ must be non-integer. We assume $p'/q' = \mathrm{pr}_1 A(p_1/q, p_2/q) \notin \mathbb{Z}$ (otherwise, we can conjugate f with a rotation by $\pi/2$, as usual). Since $A(p_1, p_2)$ is an integer point, it follows that $q' \leq q$ (if we assume p'/q' is irreducible). Thus,
$$\mathrm{pr}_1\left(\rho(AfA^{-1})\right) = \mathrm{pr}_1(A\,\rho(f)) \subset \mathrm{pr}_1\left(\frac{p'}{q'} - \frac{1}{q}, \frac{p'}{q'} + \frac{1}{q}\right).$$
It is clear that the interval above contains no integers, so that $\tilde{A}F\tilde{A}^{-1}$ (and, consequently, F) has a free curve by Theorem 2.19.
- $\rho(f)$ is a segment of irrational slope with an integer point. Then F has a fixed point by Theorem 2.8 (see also Remark 3.5).
- $\rho(f)$ is a single point. Then either this point is integer, and F has a fixed point by Theorem 2.6 or it is not integer, and F has a free curve by Theorem 2.19.
- $\rho(f)$ is a segment of rational slope. Conjugating all the maps by an element of $\mathrm{GL}(2,\mathbb{Z})$ we may assume it is a vertical segment; and with this assumption, if both $\mathrm{pr}_1(\rho(f))$ and $\mathrm{pr}_2(\rho(f))$ contain an integer, it follows that $\rho(f)$ contains an integer point, and by Theorem A, F has either a fixed point or a free curve. On the other hand, if either of the two projections contains no integer, Theorem 2.19 implies the existence of a free curve for F as before.

This concludes the proof.

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